MARTINGALES AND BROWNIAN MOTION: FINAL EXAM
8 DECEMBER, 2015

Each question carries 10 marks. Maximum you can score is 50.
Duration of the exam: 120 minutes.

1. Let $a < b$ be real numbers and let A, B, α, β be strictly positive numbers. Set $a_k = a + \frac{A}{k^\alpha}$ and $b_k = b - \frac{B}{k^\beta}$. Assume that $a < a_k < b_k < b$ for all k. Let μ_k be the uniform distribution on $[a_k, b_k]$ and let μ be the uniform distribution on $[a, b]$. Decide for which values of A, B, α, β is

(1) $\otimes_{k\geq 1}\mu_k \ll \otimes_{k\geq 1}\mu$.

(2) $\otimes_{k\geq 1}\mu_k \perp \otimes_{k\geq 1}\mu$.

2. Suppose X is a random variable with $\mathbf{P}\{X = 2\} = \mathbf{P}\{X = 1\} = \frac{1}{5}$ and $\mathbf{P}\{X = -1\} = \frac{3}{5}$. Let W denote a standard 1-dimensional Brownian motion.

(1) Find a stopping time τ wuch that $W_\tau \stackrel{d}{=} X$. [2 points extra if you do not use additional randomness]

(2) What is $\mathbf{E}[\tau]$?

3. Let (X_t, Y_t) be a standard 2-dimensional Brownian motion started at (x, y) where $y \neq 0$. Let τ be the first time when the Brownian motion hits the real line. If $R = \sqrt{x^2 + y^2}$, then show that $\mathbf{P}\{X_\tau \in [-R, R]\} = \frac{1}{2}$.

4. Let W is a standard 1-dimensional Brownian motion and let $W_{\text{br}}(t) = W(t) - tW_1$, $0 \leq t \leq 1$, be a Brownian bridge (which can also defined as W given $W_1 = 0$).

(1) Write an expression for the probability that $\mathbf{P}\{M_1 \geq u,\ |B_1| \leq \epsilon\}$ for $u > 0$.

(2) Deduce that $\mathbf{P}\{\max_{t\leq 1} W_{\text{br}}(t) \geq u\} = e^{-2u^2}$ for $u > 0$.

5. Let $X_1, X_2, \ldots$ be i.i.d. random variables with zero mean and unit variance. For $n \geq 1$, define

$$W_n^*(t) = \begin{cases} \frac{1}{\sqrt{n}}(S_k - S_n) & \text{if } t = \frac{k}{n},\ 0 \leq k \leq n, \\ \text{linear in each interval} & (\frac{k}{n}, \frac{k+1}{n}),\ 0 \leq k \leq n-1. \end{cases}$$

Show that W_n^* converges in distribution to Brownian bridge.

6. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbf{P}\{X_1 = +1\} = p$ and $\mathbf{P}\{X_1 = -1\} = q$ where $\frac{1}{2} < p < 1$ and $q = 1 - p$. Let $S_n = X_1 + \ldots + X_n$.

(1) If $\lambda > 0$, set $X_n^\lambda = \lambda^{S_n}$. For which values of λ is X^λ a sub-martingale or super-martingale or martingale?

(2) Use the first part to find the probability that the random walk $(S_n)_{n \geq 0}$ never hits the level -1.